OPTIMUM TRANSMISSION RADII FOR PACKET RADIO NETWORKS
or
WHY SIX IS A MAGIC NUMBER

Leonard Kleinrock and John Silvester

Department of Computer Science
University of California at Los Angeles
Los Angeles, California 90024

ABSTRACT

In this paper we study the capacity of packet radio networks in which the nodes are randomly located and which use slotted ALOHA as the access scheme, transporting messages through the network in a store and forward fashion. This random location of nodes can be thought of as representing either an arbitrary network or a snapshot of a mobile one.

We find that one of the major factors affecting the capacity is the transmission radius that the nodes use. Although using a very large transmission radius gives a high degree of connectivity, there will be much interference and a corresponding loss of channel throughput. In the extreme case where we have a completely connected network we know that the (ALOHA) capacity for the entire network is only $1/e$. We can limit this interference and increase the capacity by reducing the transmission radius, but doing this implies a corresponding increase in the number of hops a message must take in order to arrive at its destination. This increased number of hops creates more internal traffic which tends to reduce the effective capacity of the network.

We analyze this tradeoff and find that there is a transmission radius which optimizes the capacity and that this radius allows us to achieve a throughput proportional to the square root of the number of nodes in the network.

1. Introduction

One of the major problems in effective utilization of computer resources is the distribution of those resources to the user. This problem has been greatly alleviated by the advent of communication networks but local distribution still remains a problem. The concept of broadcast packet radio for local access was first utilized in the ALOHA system [ABRA 70] and more recently, the Advanced Research Projects Agency of the Department of Defense has undertaken a project to investigate the use of more general broadcast packet radio systems [KAHN 75]. A packet radio network consists of many packet radio units sharing a common radio channel such that when one unit transmits, many other units will hear the packet, even though it is addressed to only one of them. This feature, inherent in broadcast systems, in conjunction with the fact that we have no control over access to the channel, results in destructive interference when several packets are received simultaneously.

Many studies have been made on the capacity of single hop communication networks using broadcast radio as the communication medium. In [LAM 74] we find an extensive analysis for the one hop slotted ALOHA access scheme and in [TOBA 74, KLEI 75b] we find similar results for the Carrier Sense Multiple Access (CSMA) scheme. When we start to consider larger networks which require repeaters to forward messages (as we only have line of sight communication) we find many much more serious problems. These problems are accentuated if the network has mobile components, which was one of the major motivations for using broadcast technology. Little work exists, however, on the performance of networks requiring repeaters and store and forward packet switching. In this paper we study the capacity of such multi-hop networks and find that one of the most important factors affecting this capacity is the transmission radius used by the nodes (this corresponds to transmitter power). In a real system the power is limited and we are not in general able to reach our destination (for a large network) in one hop. In this paper, we show that this power limitation is, in fact, a desirable feature and that we may even want to reduce the transmission power to a level lower than the maximum determined by the transmitter characteristics.

Kleinrock [KLEI 75a] also found that a critical radius exists when trying to minimize delay in an arbitrary point to point network and Akavia [AKAV 78] finds similar results in trying to minimize the cost of the network for a certain delay requirement. Both of these authors assume a continuum of sources (repeaters) throughout the network, the consequence being that a transmission will always progress toward the destination by a distance equal to the transmission radius. For small transmission radii (or sparse networks) this assumption is invalid and we must take the topology into consideration. We are unable to progress to the edge of our transmission radius for two reasons: firstly, the probability of finding a point close to the edge of our transmission radius decreases as the expected number of points within range is reduced; and secondly, the probability of finding someone in the direction in which we wish to travel is also diminished.

Rather than restrict ourselves to certain specific topologies (regular networks, for example), we will consider networks, consisting of a set of nodes *randomly* located in the plane. We consider these to be nodes in a distributed (i.e. not centralized) communication network. Such a network can be thought of as either repesenting a snapshot of a mobile network or as a representative sample of the set of all networks.

We presume the existence of a routing algorithm which allows packets to be forwarded from source to destination through the network. Each packet radio unit is assumed to use a predetermined fixed radius for transmission (which determines the network structure). The performance of the network will then be studied as the transmission radius is varied. Clearly if the transmission radius is too small some of the nodes will become isolated. In this paper we restrict ourselves to consider only connected networks, however (we look at disconnected communication to some degree in an another paper [SILV 79]). By requiring that the transmission radius be large, we can make the

This research was supported by the Advanced Research Projects Agency of the Department of Defense under contract MDA 903-77-C-0272.

probability of the network not being connected small.

As we increase the transmission radius we find that the degree of connectivity increases, each node being able to communicate with more nodes in one hop. In addition to varing the transmission radius we have an additional degree of freedom, namely the transmission probability. It will be necessary to reduce the transmission probability as the connectivity increases so that the environment around any node is not overloaded with traffic. In the following analysis we will optimize this transmission probability to give the best throughput. Varying the transmission radius has two effects on the throughput of the network.

Firstly, as each node hears more other nodes, we might assume that there would be a corresponding increase in the amount of traffic received at each node. This is not true, however, since a well designed system will adjust itself (by means of the transmission probability) so that there is no traffic overload. It is true, though, that a greater proportion of the traffic heard by a node will not be addressed to him and, more importantly, he will not be allowed to transmit as frequently, thus reducing the amount of traffic which he can offer to the network.

Secondly, as if to counteract this degradation in performance, as the transmission radius increases the source-destination path lengths will be reduced as each hop will take you closer to your destination. Thus each message will require less transmissions to reach its destination and cause less load on the network, which allows a correspondingly higher traffic level to be supported.

There is clearly a tradeoff between these two and it is not at all apparent which effect will dominate. In fact we find that there is an optimum transmission radius which maximize the obtainable throughput.

2. General Model

The nodes of the network are considered to be uniformly distributed in (two-dimensional) space with density λ (that is, there will be an average of λ points per unit area). The access mode that will be used is slotted ALOHA with each node having a transmission probability p in a slot. The slots correspond to the transmission time of the longest packet used in the system and the maximum throughput in any local environment is therefore one packet per slot (although, of course, we cannot attain a throughput of one for an ALOHA system due to interference). Each node will transmit with the same radius r, which will determine the connectivity (topology) of the network. Any nodes falling within the circle of radius r about a node will be able to hear that node and also be able to transmit to it. We only consider the heavy traffic case, in which every node is always busy and will transmit whenever permitted (the restraint being the transmission probability).

The traffic matrix we will study is uniform, each node wishing to communicate with all others on an equal basis. We will therefore consider each node to be equivalent, having the same transmission radius, transmission probability and traffic load. (We are assuming here that the edge effects and imbalance of traffic due to routing are of minor importance.)

We will find the capacity of the network, which is the maximum achievable throughput measured in terms of source destination messages. We start by studying the number of transmissions per unit time that can be handled by the network.

3. Per-Hop Traffic

Consider the number of successful transmissions per slot. This is a measure of the throughput if nodes are only talking to their neighbors. If, however, some traffic requires more than one hop, we will be counting each transmission along the path as a contribution to the throughput.

Consider an arbitrary node in the network. We define h_i to be the probability of hitting i other nodes by a transmission and H_i to be the probability of being in range of i other nodes. As the nodes are randomly distributed, the number of nodes that will be in a circle of radius r is Poisson distributed, i.e.,

$$h_i = \frac{(\lambda A)^i e^{-\lambda A}}{i!} \qquad (i=0,1,2,\ldots) \tag{1}$$

where $A = \lambda\pi r^2$ is the area (volume) covered by the transmission.

We will find that the term λA continually crops up in our equations. This corresponds to the expected number of nodes in a transmission radius about any point. For convenience, therefore, let us define N to be this average degree.

$$\begin{aligned} N &= \lambda A \\ &= \lambda\pi r^2 \end{aligned} \tag{2}$$

We can therefore rewrite (1) in these terms.

$$h_i = \frac{N^i e^{-N}}{i!} \qquad (i=0,1,2,\ldots) \tag{3}$$

In the case where all nodes are using the same transmission radius, it is clear that you will hear precisely those nodes that hear you and, thus, H will have the same distribution as h, that is,

$$H_i = \frac{N^i e^{-N}}{i!} \qquad (i=0,1,2,\ldots) \tag{4}$$

We are interested in counting the number of successful transmissions in any slot. Let us, therefore, define q to be the probability of a node successfully receiving a packet in a slot, and q_i to be the same conditioned on the fact that this node hears i people. This is the probability that exactly one of the units that you hear transmits to you and you are silent. In slotted ALOHA these events are independent as there is no control and all nodes are considered constantly busy for the heavy traffic case. For simplicity let us assume that every node in the network uses the same transmission probability p. Let us define the event A_i to represent a node hearing i other nodes.

We then have:

$$\begin{aligned} q_i &= Pr\{\text{a neighbor transmits to you and you do not transmit} \mid A_i\} \\ &= Pr\{\text{exactly one of the } i \text{ units transmits} \mid A_i\} \\ &\qquad {}^*Pr\{\text{addressed to you} \mid A_i\} \, {}^*Pr\{\text{you do not transmit} \mid A_i\} \\ &= \binom{i}{1} p\,(1-p)^{i-1} \frac{1}{i}\,(1-p) \\ &= p(1-p)^i \end{aligned} \tag{5}$$

If we now uncondition on the number heard we can obtain the probability, s, of successfully receiving a packet in any particular slot.

$$s = \sum_{i=1}^{\infty} H_i q_i$$

$$= \sum_{i=1}^{\infty} \frac{N^i e^{-N} p(1-p)^i}{i!} \quad (6)$$

Summing and rewriting we obtain,

$$s = pe^{-Np} - pe^{-N} \quad (7)$$

The e^{-N} in the second term corresponds to the probability of there being nobody in range. As we are only considering connected networks we will need an average degree large enough to ensure against this. Erdos and Renyi [ERDO 59] have considered the issue of connectivity for large random graphs (i.e. graphs not defined by a geometrical relationship) and found that if the average degree is $\log(n)+c$ then the probability of the graph being connected is $e^{-e^{-c}}$. The graphs that we are interested in, however, are Euclidean graphs where the existence of edges is not an independent process. The analysis of connectivity is much more complex and no simple results like those for random graphs are known. Dewitt [DEWI 77] finds a lower bound on the probability of connectedness. If the average degree is $4\log(n) + 4\log\log(n) + 4c$ then $\Pr\{\text{connected}\} \geq e^{-e^{-c}}$. He also suggests that $\log(n)+O(\log\log(n))$ should be sufficient for connectivity. These results are asymptotically true for large graphs and may or may not be exact for smaller graphs. In Table 1 we give the average degree necessary (based on these formulae) for the probability of connectedness to be 0.95. (We have found in our simulations that using an average degree of 5 we have always been able to generate a connected network in one or two tries for networks with less than 100 nodes.)

#nodes	*Av. Deg. (Erdos)*	*Av. Deg. (DeWitt)*
10	5.2	24
20	6.0	28
40	6.6	32
80	7.3	35
150	8.0	38

Table 1 Number of edges required for connectivity (prob=.95)

We see, therefore, that we will need a degree of at least four to have a connected network. From stability arguments (so that we do not overload the local channel) [LAM 74] we know that p must decrease as N increases and in fact, should be proportional to $1/N$. The second term in Eq. 7 then becomes negligible compared to the first.

Rewriting, we obtain:

$$s = pe^{-Np} \quad (8)$$

Optimizing for p we find:

$$\frac{ds}{dp} = e^{-Np} - Npe^{-Np}$$

$$= e^{-Np}(1 - Np) = 0 \quad (9)$$

Thus:

$$p_{opt} = \frac{1}{N} \quad (10)$$

Substituting this value back into Eq. 7 we see that for a connected net ($N>4$) our assumption to neglect the second term appears to be justified.

Which gives the local throughput s, (i.e. throughput per node):

$$s = \frac{1}{Ne} \quad (11)$$

The fact that the optimum value of p is found to be $1/N$ is no surprise as it corresponds to setting the average traffic load G to be equal to one packet per slot in any local environment [ABRA 70, LAM 74].

4. Network Utilization

From s we can determine the expected number of successful transmissions per slot for the whole network, s_{net}, by multiplying by the total number of nodes n:

$$s_{net} = \frac{n}{Ne} \quad (12)$$

If we set N to be equal to n, which is equivalent to allowing all nodes to hear each other (i.e. very large transmission radius), the throughput reduces to $1/e$ which is Abramson's result for such nets [ABRA 70] (the path lengths are 1 in this case).

5. Network Throughput

The quantity obtained above is a measure of the number of successful transmissions per slot. In the transit of an arbitrary message from source to destination, it will in general be transmitted several times as it threads its way along the path defined by the routing matrix. The above computations will count each of these transmissions as a contribution to the throughput. We need therefore to divide the above figure by the average path length of the network. We will then have a measure of the true network throughput in terms of messages delivered.

Clearly this average path length is traffic matrix dependent. In fact, if we consider a traffic matrix which only specifies nearest neighbor communication, we have that the number of successful transmissions is indeed equivalent to the network throughput. This is not the interesting case, however. We will consider the more general case in which we assume that each node wishes to communicate with every other node in the network on an equal basis. That is to say, for each message generated at a node, we will randomly select the destination from the set of other nodes in the network. This is a uniform traffic matrix.

We need therefore to find the traffic-weighted path length, which, for the uniform traffic matrix, is the same as the usual concept of path length in a graph. The determination of average path length in a random graph is hard, and so we proceed by calculating the expected progress per hop. If the points were infinitely dense (compared to the transmission radius) we would expect to always be able to reach the edge of our transmission range in any direction in which we wished to travel. As the radius decreases however we will find that the point which will allow us to make the most progress towards our destination will be further and further away from the circumference. Eventually, in fact, we will not be able to make any progress at all in the direction we wish (the graph is likely to be disconnected by this time).

Dividing the expected distance between a random pair of points in the graph by the expected progress in one hop, we find the expected number of hops to reach an arbitrary destination. This is equivalent to the average path length.

6. Expected Progress

Let us consider the expected progress in one hop, z. In Figure 1, P is the source having a message destined to Q (in fact P can also be one of the intermediate points along the path defined by the routing matrix). Any point on the arc centered at Q is equivalent in terms of progress, the distance z is then measured from P to this arc.

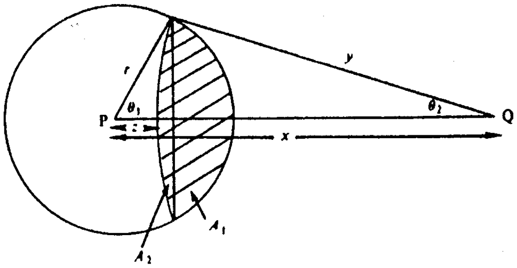

Figure 1. Progress in One Hop

Let us define:

$$F(z) = Pr\{Z \leqslant z\}$$

$$= e^{-\lambda A} \qquad (13)$$

where A is the shaded area.

A is composed of two spherical caps A_1 and A_2:

$$A_1 = r^2\left[\theta_1 - \frac{\sin(2\theta_1)}{2}\right] \qquad A_2 = y^2\left[\theta_2 - \frac{\sin(2\theta_2)}{2}\right]$$

where the angles are given by:

$$\theta_1 = \cos^{-1}\left[\frac{r^2+x^2-y^2}{2rx}\right] \qquad \theta_2 = \cos^{-1}\left[\frac{y^2+x^2-r^2}{2xy}\right]$$

If P is sufficiently distant from Q we may neglect A_2 and for convenience, in the following we only consider A_1. We in fact did study the effect of including the correction term of A_2 and found that it made no significant difference to the average path length computation.

We can thus find $\bar{z}$ (the expected progress) to be

$$\bar{z} = \int_0^r [1 - F(x)]\, dx - \int_{-r}^{0} F(x) dx + re^{-\lambda\pi r^2}$$

The last term in this expression corresponds to the probability of nobody being in range, and the second integral corresponds to the case where no progress can be made (i.e. we must move away from our destination). It could be argued that this term should not be included (depending on the routing strategy used), but we include it for completeness in the geometrical argument. It will have a negligible contribution to the computation for the range of degrees that we shall consider (i.e. those that will guarantee connectivity). Making the substitution $t = \cos(\theta_1)$ we have:

$$\bar{z} = r\left[1 + e^{-\lambda\pi r^2} - \int_{-1}^{1} e^{-\lambda r^2\left[\cos^{-1}(t) - t\sqrt{1-t^2}\right]} dt\right]$$

$$= \left(\frac{N}{\lambda\pi}\right)^{\frac{1}{2}}\left[1 + e^{-N} - \int_{-1}^{1} e^{-\frac{N}{\pi}\left[\cos^{-1}(t) - t\sqrt{1-t^2}\right]} dt\right] \qquad (14)$$

If we consider the progress factor (normalized with respect to the radius) $f = z/r$, we find that it is a function depending only on N rather than explicitly on the radius. Figure 2 shows the expected progress factor as a function of the expected degree N. Although we show the curve for small values of N, the curve probably does not represent the true progress that would be made in a real network, due to connectivity limitations and that the routing procedure may not allow to move away from our destination.

7. Expected Path Length

In order to determine the average path length we need to find the average distance between any two points in the network. This is equivalent to finding the distance between two points randomly located inside the area in which the network is enclosed. If we assume that the network is situated inside a disc of radius R, then the expected distance, d, between any two points randomly located within this disc is given by [KEND 63].

$$d = \frac{128}{45\pi} R \qquad (15)$$

We need to express R in terms of the density and total number of nodes.

$$\lambda\pi R^2 = n$$

$$\Rightarrow R = \left(\frac{n}{\lambda\pi}\right)^{\frac{1}{2}} \qquad (16)$$

We can, thus, find the average number of hops $\bar{h}$ to be:

$$\bar{h} = \frac{d}{\bar{z}} = \frac{128}{45\pi}\left(\frac{n}{N}\right)^{\frac{1}{2}} \frac{1}{1 + e^{-N} - \int_{-1}^{1} e^{-\frac{N}{\pi}\left[\cos^{-1}(t) - t\sqrt{1-t^2}\right]} dt} \qquad (17)$$

8. Network Throughput

We can now determine the true network throughput, γ, by dividing the number of successful transmissions (Eq. 12) by the number of times a packet is repeated (the average path length given in Eq. 17).

$$\gamma = \frac{45\pi}{128e}\left(\frac{n}{N}\right)^{\frac{1}{2}}\left[1 + e^{-N} - \int_{-1}^{1} e^{-\frac{N}{\pi}\left[\cos^{-1}(t) - t\sqrt{1-t^2}\right]} dt\right] \qquad (18)$$

This equation is the main result of this paper, showing the network throughput as a function of the average degree. It expresses the tradeoff between small transmission radii (many hops) and large transmission radii (too much interference). If the average degree is a constant we see that the throughput is proportional to the square root of the number of nodes in the network. If the degree is an increasing function of the number of nodes however, the capacity will grow at a rate slower than $\sqrt{n}$. We show in Fig. 3 the normalized network throughput $\frac{\gamma}{\sqrt{n}}$. The value of N which maximizes the throughput is 5.89, at which point the optimal network througphut γ^* is given by:

$$\gamma^* = .0976\sqrt{n} \tag{19}$$

which should be compared to the ALOHA (fully-connected) througphut of $1/e$ independent of the network size. We also notice that the throughput is extremely sensitive to reduction in degree from this optimum, whereas the capacity is relatively insensitive to the use of larger degrees.

Fig. 4 shows the network throughput given by Eq. 18 as a function of the number of nodes, for various average degrees. For comparison purposes we show the curve for a completely connected ALOHA network which is asymptotic to $1/e$ for large nets (slightly exceeding this for small nets [ABRA 70]). The curves for γ^* are only valid for average degrees greater than the network size, as the performance reduces to that of the completely connected net for degrees close to the number of nodes. The reason that Eq. 18 is not valid for average degrees comparable to the network size is that we must use a more sophisticated computation for path length to incorporate edge effects and the area A_2 mentioned in section 6.

9. Conclusions

We have shown that for a constant average degree in a random network we can obatih a throughput proportional to the square root of the number of nodes on the network. We have also shown that the optimal average degree is approximately 6. Using a degree less than 6 causes drastic reduction in capacity of the network (the network also becoming disconnected), whereas exceeding 6 causes only gradual degradation (provided we do not have a degree which is an increasing function of the number of nodes). When an average degree of 6 is used the network throughput is $.0976\sqrt{n}$, as opposed to $1/e$ for a fully connected network.

References

ABRA 70 Abramson, N., "The ALOHA System - Another Alternative for Computer Communications", *AFIPS Conference Proceedings,* 1970 Fall Joint Computer Conference, Vol 37, pp 281-285.

AKAV 78 Akavia, G.Y., "Hierarchical Organization of Distributed Packet-Switching Communication Systems", Ph.D. Dissertation, 1978, Department of Computer Science, U.C.L.A., Los Angeles.

DEWI 77 Dewitt, H., "The Theory of Random Graphs with Applications to the Probabilistic Analysis of Optimization Algorithms", Ph.D. Dissertation, 1977, Department of Computer Science, U.C.L.A., Los Angeles.

ERDO 59 Erdos, P. and Renyi, A. "On Random Graphs I", *Publications Mathematicae,* Debrecen 6 1959, pp 290-297.

KAHN 75 Kahn, R.E., "The Organization of Computer Resources into a Packet Radio Network", *AFIPS Conference Proceedings Vol 44,* 1975, pp 177-186.

KEND 63 Kendall, M.G. and Moran, P.A.P., *Geometrical Probability,* Griffin, London, 1963.

KLEI 75a Kleinrock, L., "On Giant Stepping in Packet Radio Networks", *UCLA Packet Radio Temporary Note #5,* PRT 136, March 1975.

KLEI 75b Kleinrock, L., and Tobagi, F.A., "Packet Switching in Radio Channels: Part I--Carrier Sense Multiple Access Modes and Their Throughput Delay Characteristics", *IEEE Transactions on Communications,* Vol COM-23, No. 12, December 1975, pp 1400-1416.

LAM 74 Lam, S., "Packet Switching in a Multi-Access Broadcast Channel with Application to Satellite Communication in a Computer Network", School of Engineering and Applied Science, University of California, Los Angeles, UCLA-ENG-7429, April 1974.

SILV 79 Silvester, J.A.., and L.Kleinrock, "Capacity of Neighbour Communication in Packet Radio Networks", 12th. Hawaii International Conference on System Sciences 1979 (to appear).

TOBA 74 Tobagi, F.A., "Random Access Techniques for Data Transmission over Packet Switched Radio Networks", Ph. D. Dissertation, U.C.L.A. Department of Computer Science, UCLA-ENG-7499.

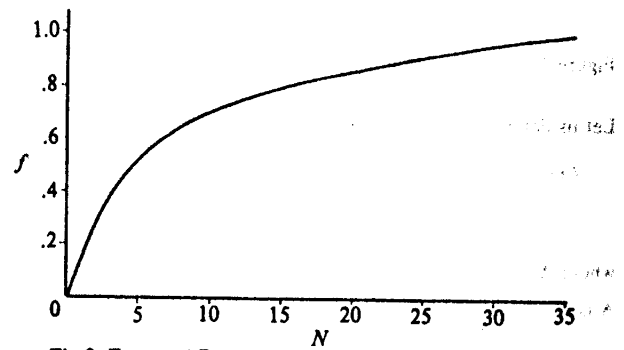

Fig 2. Expected Progress as a Function of Average Degree

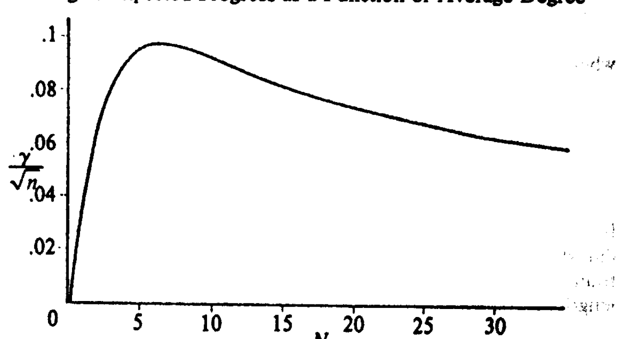

Fig 3. Normalized Network Throughput as a Function of Average Degree

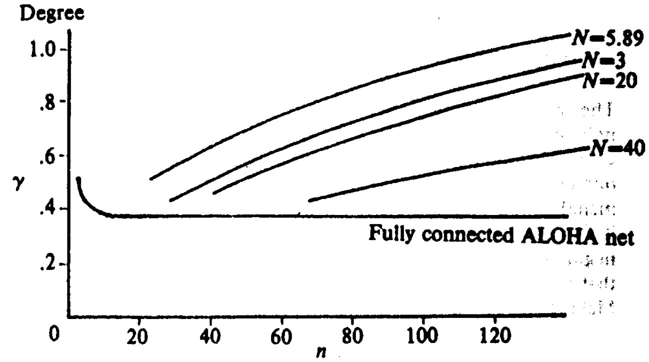

Fig 4. Network Capacity as a Function of the Number of Nodes